14 Autoregressive Moving Average Models

In this chapter an important parametric family of stationary time series is introduced, the family of the autoregressive moving average, or ARMA, processes. For a large class of autocovariance functions $\gamma(\cdot)$ it is possible to find an ARMA process $\{X_t\}$ with ACVF $\gamma_X(\cdot)$ such that $\gamma(\cdot)$ is well approximated by $\gamma_X(\cdot)$. In particular, for any positive integer K, there exists an ARMA process $\{X_t\}$ such that $\gamma_X(h) = \gamma(h)$ for $h = 0, 1, \dots, K$. For this (and other) reasons, the family of ARMA processes plays a key role in the modeling of time series data. The linear structure of ARMA processes also leads to a substantial simplification of the general methods for linear prediction (see Chapter 15).

Example. Figure 14.1 shows different ARMA processes with the corresponding autocorrelation function and partial autocorrelation function (see Section 14.3).

14.1 ARMA(1, 1) Processes

We start with an ARMA(1, 1) process to introduce some key properties of the autoregressive moving average processes (ARMA processes).

Definition 14.1.1. The time series $\{X_t\}$ is an ARMA(1, 1) process if it is stationary and for every t satisfies

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1},$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and $\phi + \theta \neq 0$.

Proposition 14.1.2. *A stationary solution of the* ARMA(1, 1) *equation exists if and only if* $\phi \neq \pm 1$.

- *If* $|\phi| < 1$, *then the unique stationary solution is given by the* MA(∞) *process*

  $$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}.$$

  In this case $\{X_t\}$ *is called causal (or future-independent) or a causal function of* $\{Z_t\}$, *since* X_t *can be expressed in terms of the current and past values* Z_s, $s \leq t$.

- *If* $|\phi| > 1$, *then the unique stationary solution is*

  $$X_t = -\theta \phi^{-1} Z_t - (\phi + \theta) \sum_{j=1}^{\infty} \phi^{-j-1} Z_{t+j}.$$

  The solution is noncausal, since $\{X_t\}$ *is a function of* Z_s, $s \geq t$.

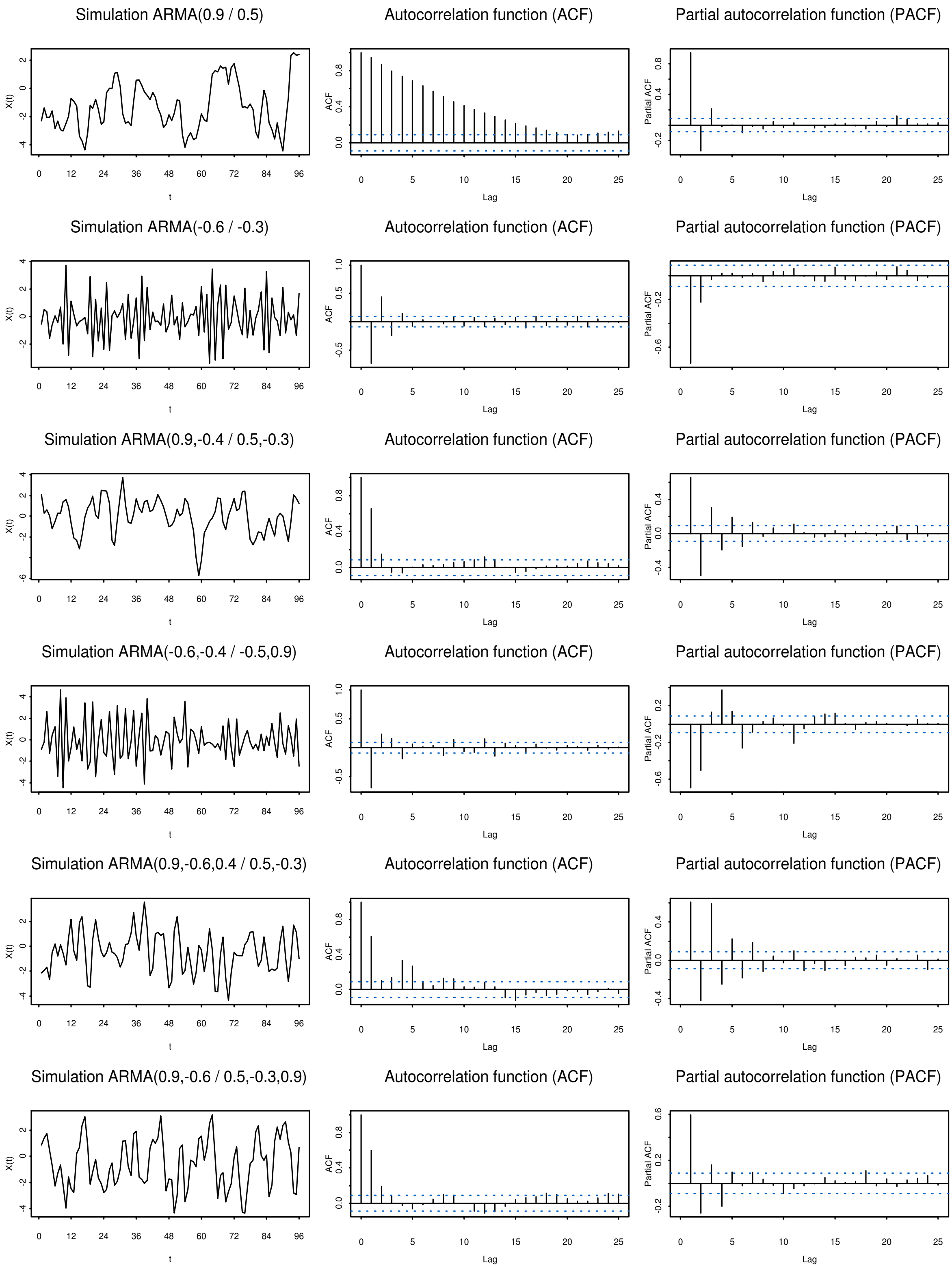

Figure 14.1: Simulations of different ARMA processes. The left column shows an excerpt ($m = 96$) of the whole time series ($n = 480$).

Example. The processes $X_t - \phi X_{t-1} = Z_t$ with $|\phi| > 1$ are called explosive, because the values of the time series quickly become large in magnitude.

- However, it is possible to modify this time series to obtain a stationary process as follows. Write $X_{t+1} = \phi X_t + Z_{t+1}$, in which case

$$\begin{aligned} X_t &= \phi^{-1}X_{t+1} - \phi^{-1}Z_{t+1} = \phi^{-1}(\phi^{-1}X_{t+2} - \phi^{-1}Z_{t+2}) - \phi^{-1}Z_{t+1} \\ &\vdots \\ &= \phi^{-k}X_{t+k} - \sum_{j=1}^{k} \phi^{-j} Z_{t+j} \end{aligned}$$

  by iterating forward k steps. Because $|\phi|^{-1} < 1$, this result suggests the stationary future dependent AR(1) model

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}.$$

  Unfortunately, this model is useless because it requires us to know the future to be able to predict the future, i.e. X_t is noncausal.

- Nevertheless, excluding explosive models from consideration is not a problem because the models have causal counterparts. For example the two processes

$$\begin{aligned} X_t - \phi X_{t-1} &= Z_t, \qquad \text{with} \quad |\phi| > 1 \text{ and } \{Z_t\} \sim \text{IID N}(0, \sigma_Z^2), \\ Y_t - \phi^{-1} Y_{t-1} &= W_t, \qquad \text{with} \quad \{W_t\} \sim \text{IID N}(0, \sigma_Z^2 \phi^{-2}) \end{aligned}$$

  are stochastically equal, i.e. all finite distributions of the processes are the same. For example, if $X_t - 2X_{t-1} = Z_t$ with $\sigma_Z^2 = 1$, then $Y_t - \frac{1}{2}Y_{t-1} = W_t$, with $\sigma_W^2 = \frac{1}{4}$, is an equivalent causal process.

Just as causality means that X_t is expressible in terms of Z_s, $s \le t$, the dual concept of invertibility means that Z_t is expressible in terms X_s, $s \le t$.

Proposition 14.1.3. *The* ARMA(1, 1) *process is*

- *invertible if* $|\theta| < 1$, *and* Z_t *is expressed in terms of* X_s, $s \le t$, *by*

$$Z_t = X_t - (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{j-1} X_{t-j},$$

- *noninvertible if* $|\theta| > 1$, *and* Z_t *is expressed in terms of* X_s, $s \ge t$, *by*

$$Z_t = -\phi\theta^{-1} X_t + (\phi + \theta) \sum_{j=1}^{\infty} (-\theta)^{-j-1} X_{t+j}.$$

14.2 ARMA(p, q) Processes

Definition 14.2.1. The time series $\{X_t\}$ is an ARMA(p, q) process if it is stationary and if for every t it satisfies

$$X_t - \phi_1 X_{t-1} - \ldots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}, \tag{14.1}$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and the polynomials

$$(1 - \phi_1 z - \ldots - \phi_p z^p)$$

and

$$(1 + \theta_1 z + \ldots + \theta_q z^q)$$

have no common factors.

Example. Consider the model $X_t - \phi X_{t-1} = Z_t - \phi Z_{t-1}$ which looks like an ARMA$(1, 1)$ process and can also be written as $(1 - \phi B)X_t = (1 - \phi B)Z_t$. Apply the operator $(1 - \phi B)^{-1}$ to both sides to obtain $X_t = Z_t$. Therefore X_t is simply a white noise process. The reason for this redundancy is the common factor in the polynomials $(1 - \phi z)$ and $(1 + \theta z)$.

Remark. It is convenient to use the form

$$\phi(B)X_t = \theta(B)Z_t,$$

where $\phi(\cdot)$ and $\theta(\cdot)$ are the pth and the qth-degree polynomials

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$$

and

$$\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q,$$

and B is the backward shift operator.

Definition 14.2.2. The process $\{X_t\}$ is said to be an

- ARMA(p, q) process with mean μ if $\{X_t - \mu\}$ is an ARMA(p, q) process,
- AR(p) process if $\theta(z) \equiv 1$ and
- MA(q) process if $\phi(z) \equiv 1$.

An important part of Definition 14.2.1 is the requirement that $\{X_t\}$ be stationary. For the ARMA$(1, 1)$ we showed in Proposition 14.1.2, that a stationary solution exists and is unique if and only if $\phi \neq \pm 1$. The analogous condition for the general ARMA(p, q) process is $\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p \neq 0$ for all complex z with $|z| = 1$. Complex z is used here, since the zeros of a polynomial of degree $p > 1$ may be either real or complex. The region defined by the set of complex z such that $|z| = 1$ is referred to as the unit circle.

Example. Consider the ARMA$(2,1)$ process $X_t - \frac{3}{4}X_{t-1} + \frac{9}{16}X_{t-2} = Z_t + \frac{5}{4}Z_{t-1}$ with $\{Z_t\} \sim \mathrm{WN}(0,\sigma^2)$. The polynomial $\phi(z) = 1 - \frac{3}{4}z + \frac{9}{16}z^2$ has zeros at $z_{1,2} = 2(1 \pm i\sqrt{3})/3$ which lie outside the unit circle. The process therefore is causal. On the other hand, the polynomial $\theta(z) = 1 + \frac{5}{4}z$ has a zero at $z = -\frac{4}{5}$, and hence the process $\{X_t\}$ is not invertible.

Proposition 14.2.3 (Existence and uniqueness)**.** *A stationary solution* $\{X_t\}$ *of* (14.1) *exists (and is also the unique stationary solution) if and only if*

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p \neq 0 \quad \textit{for all } |z| = 1.$$

Proposition 14.2.4 (Causality or future-independence)**.** *An* ARMA(p,q) *process* $\{X_t\}$ *is causal if there exist constants* $\{\psi_j\}$ *such that*

$$\sum_{j=0}^{\infty} |\psi_j| < \infty \qquad \textit{and} \qquad X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \quad \textit{for all } t. \tag{14.2}$$

Causality is equivalent to the condition

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p \neq 0 \quad \textit{for all } |z| \leq 1.$$

Remark. The sequence $\{\psi_j\}$ in (14.2) is determined by the relation

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \theta(z)/\phi(z)$$

or equivalently by

$$\psi_j - \sum_{k=1}^{p} \phi_k \psi_{j-k} = \theta_j, \quad j = 0, 1, \ldots,$$

where $\theta_0 := 1$, $\theta_j := 0$ for $j > q$, and $\psi_j := 0$ for $j < 0$.

Proposition 14.2.5 (Invertibility)**.** *An* ARMA(p,q) *process* $\{X_t\}$ *is invertible if there exist constants* $\{\pi_j\}$ *such that*

$$\sum_{j=0}^{\infty} |\pi_j| < \infty \qquad \textit{and} \qquad Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \quad \textit{for all } t. \tag{14.3}$$

Invertibility is equivalent to the condition

$$\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q \neq 0 \quad \textit{for all } |z| \leq 1.$$

Remark. The sequence $\{\pi_j\}$ in (14.3) is determined by the equations

$$\pi_j + \sum_{k=1}^{q} \theta_k \pi_{j-k} = -\phi_j, \quad j = 0, 1, \ldots,$$

where $\phi_0 := -1$, $\phi_j := 0$ for $j > p$, and $\pi_j := 0$ for $j < 0$.

Example. Consider the process

$$X_t - \frac{4}{10}X_{t-1} - \frac{9}{20}X_{t-2} = Z_t + Z_{t-1} + \frac{1}{4}Z_{t-2}$$

or, in operator form,

$$\left(1 - \frac{4}{10}B - \frac{9}{20}B^2\right) X_t = \left(1 + B + \frac{1}{4}B^2\right) Z_t.$$

At first, X_t appears to be an ARMA$(2, 2)$ process. But, the associated polynomials

$$\phi(z) = (1 + 0.5z)(1 - 0.9z)$$
$$\theta(z) = (1 + 0.5z)^2$$

have a common factor that can be canceled. So the model is an ARMA$(1, 1)$ process

$$(1 - 0.9B)X_t = (1 + 0.5B)Z_t.$$

It is causal because $(1 - 0.9z) = 0$ when $z = \frac{10}{9}$ which is outside the unit circle and also invertible because $(1 + 0.5z) = 0$ when $z = -2$ which is also outside the unit circle. The causal representation is

$$X_t = Z_t + 1.4 \sum_{j=1}^{\infty} 0.9^{j-1} Z_{t-j},$$

the invertible one is

$$X_t - 1.4 \sum_{j=1}^{\infty} (-0.5)^{j-1} X_{t-j} = Z_t.$$

Proposition 14.2.6. *Let* $\{X_t\}$ *be the* ARMA(p, q) *process satisfying the equations*

$$\phi(B)X_t = \theta(B)Z_t, \qquad \{Z_t\} \sim \text{WN}(0, \sigma^2),$$

where $\phi(z) \neq 0$ *and* $\theta(z) \neq 0$ *for all* $|z| = 1$. *Then there exist polynomials,* $\tilde{\phi}(z)$ *and* $\tilde{\theta}(z)$, *nonzero for* $|z| \leq 1$, *of degree* p *and* q *respectively, and a white noise sequence* $\{Z_t^*\}$ *such that* $\{X_t\}$ *satisfies the causal invertible equation*

$$\tilde{\phi}(B)X_t = \tilde{\theta}(B)Z_t^*.$$

Proof. Define

$$\tilde{\phi}(z) = \phi(z) \prod_{r<j\leq p} \frac{1 - a_j z}{1 - a_j^{-1} z}$$
$$\tilde{\theta}(z) = \theta(z) \prod_{s<j\leq q} \frac{1 - b_j z}{1 - b_j^{-1} z},$$

where $a_{r+1}, \ldots, a_p$ and $b_{s+1}, \ldots, b_q$ are the zeros of $\phi(z)$ and $\theta(z)$ which lie inside the unit circle. Since $\tilde{\phi}(z) \neq 0$ and $\tilde{\theta}(z) \neq 0$ for all $|z| \leq 1$, it suffices to show that the process defined by

$$Z_t^* = \frac{\tilde{\phi}(z)}{\tilde{\theta}(z)} X_t$$

is white noise, i.e.,

$$\{Z_t^*\} \sim \mathrm{WN}\left(0, \sigma^2 \left(\prod_{r<j\leq p} |a_j|^2\right) \left(\prod_{r<j\leq p} |b_k|^{-2}\right)\right).$$

□

Example. The ARMA process

$$X_t - 2X_{t-1} = Z_t + 4Z_{t-1}, \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2),$$

is neither causal nor invertible. Introducing $\tilde{\phi}(z) = 1 - 0.5z$ and $\tilde{\theta}(z) = 1 + 0.25z$, we see that $\{X_t\}$ has the causal invertible representation

$$X_t - 0.5X_{t-1} = Z_t^* + 0.25Z_{t-1}^*, \qquad \{Z_t^*\} \sim \mathrm{WN}(0, 4\sigma^2).$$

14.3 Autocorrelation and Partial Autocorrelation Function of ARMA(p,q) Processes

First we calculate the autocovariance and autocorrelation function of a causal ARMA(p,q) process $\{X_t\}$. Secondly we define the partial autocorrelation function (PACF).

14.3.1 Calculation of the Autocovariance Function

Let

$$\phi(B)X_t = \theta(B)Z_t, \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2),$$

be a causal ARMA(p,q) process. The causality assumption implies that

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \tag{14.4}$$

where

$$\sum_{j=0}^{\infty} \psi_j z^j = \theta(z)/\phi(z), \quad |z| \leq 1. \tag{14.5}$$

From Proposition 13.2.2 and (14.4) we obtain

$$\gamma(h) = \mathrm{E}(X_{t+h}X_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|}.$$

Example. The autocovariance function of an ARMA$(1, 1)$ process

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

with $|\phi| < 1$ is given by

$$\begin{aligned}
\gamma_X(0) &= \sigma^2 \sum_{j=0}^{\infty} \psi_j^2 \\
&= \sigma^2 \left[1 + \frac{(\theta + \phi)^2}{1 - \phi^2} \right], \\
\gamma_X(1) &= \sigma^2 \sum_{j=0}^{\infty} \psi_{j+1} \psi_j \\
&= \sigma^2 \left[\theta + \phi + \frac{(\theta + \phi)^2 \phi}{1 - \phi^2} \right],
\end{aligned}$$

and

$$\gamma_X(h) = \phi^{h-1} \gamma(1), \quad h \geq 2.$$

The calculation of the autocorrelation function is straightforward

$$\rho_X(h) := \frac{\gamma(h)}{\gamma(0)}.$$

14.3.2 Partial Autocorrelation Function

The partial autocorrelation function, like the autocorrelation function, conveys information regarding the dependence structure of a stationary process. The partial autocorrelation $\alpha(k)$, $k \geq 2$, is the correlation of the two residuals obtained after regressing X_{k+1} and X_1 on the intermediate observations $X_2, \ldots, X_k$.

Example. To motivate the idea of partial autocorrelation function consider the causal AR(1) model, $X_t - \phi X_{t-1} = Z_t$. Then,

$$\gamma_X(2) = \text{Cov}(X_t, X_{t-2}) = \text{Cov}(\phi^2 X_{t-2} + \phi Z_{t-1} + Z_t, X_{t-2}) = \phi^2 \gamma(0).$$

Suppose we break this chain of dependence by removing the effect X_{t-1}. That is, we consider the correlation between $X_t - \phi X_{t-1}$ and $X_{t-2} - \phi X_{t-1}$, because it is the correlation between X_t and X_{t-2} with the liner dependence of each on X_{t-1} removed. In this way, we have broken the dependence chain between X_t and X_{t-2}. In fact,

$$\text{Cov}(X_t - \phi X_{t-1}, X_{t-2} - \phi X_{t-1}) = \text{Cov}(Z_t, X_{t-2} - \phi X_{t-1}) = 0.$$

Hence, the tool we need is partial autocorrelation, which is the correlation between X_t and X_s with the linear effect of everything "in the middle" removed.

To formally define the partial autocorrelation function for a mean-zero stationary time series, let $\hat{X}_{t+h}$, for $h \geq 2$, denote the regression of X_{t+h} on $\{X_{t+h-1}, X_{t+h-2}, \ldots, X_{t+1}\}$, which we write as

$$\hat{X}_{t+h} = \beta_1 X_{t+h-1} + \beta_2 X_{t+h-2} + \cdots + \beta_{h-1} X_{t+1}. \tag{14.6}$$

No intercept is needed because the mean of X_t is zero. In addition, let $\hat{X}_t$ denote the regression of X_t on $\{X_{t+1}, X_{t+2}, \ldots, X_{t+h-1}\}$, then

$$\hat{X}_t = \beta_1 X_{t+1} + \beta_2 X_{t+2} + \cdots + \beta_{h-1} X_{t+h-1}. \tag{14.7}$$

Because of stationarity, the coefficients $\beta_1, \ldots, \beta_{h-1}$ are the same in (14.6) and (14.7).

Definition 14.3.1. The partial autocorrelation function $\alpha(\cdot)$ of a stationary time series is defined by

$$\alpha(1) = \mathrm{Cor}(X_2, X_1) = \rho(1),$$

and

$$\alpha(k) = \mathrm{Cor}(X_{k+1} - \hat{X}_{k+1}, X_1 - \hat{X}_1), \quad k \geq 2.$$

Proposition 14.3.2. *An equivalent definition of the partial autocorrelation function on an ARMA process $\{X_t\}$ is the function $\alpha(\cdot)$ defined by*

$$\alpha(0) = 1$$

and

$$\alpha(h) = \phi_{hh}, \quad h \geq 1,$$

where ϕ_{hh} is the last component of

$$\boldsymbol{\phi}_h = \boldsymbol{\Gamma}_h^{-1} \boldsymbol{\gamma}_h, \tag{14.8}$$

$\boldsymbol{\gamma}_h(1) = (\gamma(1), \ldots, \gamma(h))'$ *and* $\boldsymbol{\Gamma}_h = [\gamma(i-j)]_{i,j=1}^h$.

Example. For MA(1) processes, it can be shown from (14.8) that the partial autocorrelation function at lag h is

$$\alpha(h) = \phi_{hh} = \frac{-(-\theta)^h}{(1 + \theta^2 + \cdots + \theta^{2h})}.$$

Let lag $h = 2$. Recall from (13.6) that

$$\gamma(0) = \sigma^2(1+\theta^2) \quad \text{and} \quad \gamma(1) = \sigma^2\theta \quad \text{and} \quad \gamma(2) = 0.$$

It follows that

$$\boldsymbol{\Gamma}_2 = \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\Gamma}_2^{-1} = \frac{1}{\gamma^2(0) - \gamma^2(1)} \begin{pmatrix} \gamma(0) & -\gamma(1) \\ -\gamma(1) & \gamma(0) \end{pmatrix}$$

Then

$$\boldsymbol{\phi}_2 = \boldsymbol{\Gamma}_2^{-1} \boldsymbol{\gamma}_2 \qquad \text{and} \qquad \alpha(2) = \frac{-\theta^2}{1 + \theta^2 + \theta^4}.$$

Example. For causal AR(p) processes the best linear predictor of X_{h+1} in terms of $1, X_1, \ldots, X_h$ is

$$\hat{X}_{h+1} = \phi_1 X_h + \phi_2 X_{h-1} + \ldots + \phi_p X_{h+1-p}.$$

Since the coefficient ϕ_{hh} of X_1 is ϕ_p if $h = p$ and 0 if $h > p$, we conclude that the partial autocorrelation function $\alpha(\cdot)$ of the process $\{X_t\}$ has the properties

$$\alpha(p) = \phi_p$$

and

$$\alpha(h) = 0, \qquad \text{for } h > p.$$

For $h < p$ the values of $\alpha(h)$ can easily be computed from (14.8).