13 Stationary Processes

In time series analysis our goal is to predict a series that typically is not deterministic but contains a random component. If this random component is stationary, then we can develop powerful techniques to forecast its future values. These techniques will be developed and discussed in this chapter.

13.1 Basic Properties

In Section 12.4 we introduced the concept of stationarity and defined the autocovariance function (ACVF) of a stationary time series $\{X_t\}$ at lag h as

$$\gamma_X(h) = \text{Cov}(X_{t+h}, X_t), \quad h = 0, \pm 1, \pm 2 \ldots$$

and the autocorrelation function as

$$\rho_X(h) := \frac{\gamma_X(h)}{\gamma_X(0)}.$$

The autocovariance function and autocorrelation function provide a useful measure of the degree of dependence among the values of a time series at different times and for this reason play an important role when we consider the prediction of future values of the series in terms of the past and present values. They can be estimated from observations of $X_1, \ldots, X_n$ by computing the sample autocovariance function and autocorrelation function as described in Definition 12.4.4.

Proposition 13.1.1. *Basic properties of the autocovariance function $\gamma(\cdot)$:*

$$\begin{aligned} \gamma(0) &\geq 0, \\ |\gamma(h)| &\leq \gamma(0) \text{ for all } h, \\ \gamma(h) &= \gamma(-h) \text{ for all } h, \text{ i.e., } \gamma(\cdot) \text{ is even.} \end{aligned}$$

Definition 13.1.2. A real-valued function κ defined on the integers is non-negative definite if

$$\sum_{i,j=1}^{n} a_i \kappa(i-j) a_j \geq 0$$

for all positive integers n and vectors $\boldsymbol{a} = (a_1, \ldots, a_n)'$ with real-valued components a_i.

Proposition 13.1.3. *A real-valued function defined on the integers is the autocovariance function of a stationary time series if and only if it is even and non-negative definite.*

Proof. To show that the autocovariance function $\gamma(\cdot)$ of any stationary time series $\{X_t\}$ is non-negative definite, let $\boldsymbol{a}$ be any $n \times 1$ vector with real components $a_1, \ldots, a_n$ and let $\boldsymbol{X}_n = (X_1, \ldots, X_n)'$. By the non-negativity of variances,

$$\text{Var}(\boldsymbol{a}'\boldsymbol{X}_n) = \boldsymbol{a}'\boldsymbol{\Gamma}_n\boldsymbol{a} = \sum_{i,j=1}^{n} a_i \gamma(i-j) a_j \geq 0,$$

where $\mathbf{\Gamma}_n$ is the covariance matrix of the random vector $\boldsymbol{X}_n$. The last inequality, however, is precisely the statement that $\gamma(\cdot)$ is non-negative definite. The converse result, that there exists a stationary time series with autocovariance function κ if κ is even, real-valued, and non-negative definite, is more difficult to establish (details see Brockwell and Davis (1991)). $\square$

Example. Let us show that the real-valued function $\kappa(\cdot)$ defined on the integers by

$$\kappa(h) = \begin{cases} 1 & \text{if } h = 0, \\ \rho & \text{if } h = \pm 1, \\ 0 & \text{otherwise}, \end{cases}$$

is an autocovariance function of a stationary time series if and only if $|\rho| \leq 1/2$.

- If $|\rho| \leq 1/2$ then $\kappa(\cdot)$ is the autocovariance function of an MA(1) process (see (12.2), p. 12-12) with $\sigma^2 = (1+\theta^2)^{-1}$ and $\theta = (2\rho)^{-1}(1 \pm \sqrt{1-4\rho^2})$.
- If $\rho > 1/2$, $\mathbf{K} = [\kappa(i-j)]_{i,j=1}^n$ and $\boldsymbol{a}$ is the n-component vector $\boldsymbol{a} = (1, -1, 1, -1, \ldots)'$, then
$$\boldsymbol{a}'\mathbf{K}\boldsymbol{a} = n - 2(n-1)\rho < 0 \quad \text{for} \quad n > \frac{2\rho}{2\rho - 1},$$
which shows that $\kappa(\cdot)$ is not non-negative definite and therefore is not an autocovariance function.
- If $\rho < -1/2$, the same argument using the n-component vector $\boldsymbol{a} = (1, 1, 1, \ldots)'$ again shows that $\kappa(\cdot)$ is not non-negative definite.

Remark. If $\{X_t\}$ is a stationary time series, then the vector $(X_1, \ldots, X_n)'$ and the time-shifted vector $(X_{1+h}, \ldots, X_{n+h})'$ have the same mean vectors and covariance matrices for every integer h and positive integer n.

Definition 13.1.4. $\{X_t\}$ is a strictly stationary time series if

$$(X_1, \ldots, X_n)' \stackrel{d}{=} (X_{1+h}, \ldots, X_{n+h})'$$

for all integers h and $n \geq 1$. Here $\stackrel{d}{=}$ is used to indicate that the two random vectors have the same joint distribution function.

Proposition 13.1.5. *Properties of a strictly stationary time series* $\{X_t\}$*:*

- *The random variable* X_t *are identically distributed;*
- $(X_t, X_{t+h})' \stackrel{d}{=} (X_1, X_{1+h})'$ *for all integers* t *and* h*;*
- $\{X_t\}$ *is weakly stationary if* $\mathrm{E}(X_t^2) < \infty$ *for all* t*;*
- *Weak stationarity does not imply strict stationarity;*

- *An iid sequence is strictly stationary.*

Definition 13.1.6 (**MA(q)** process)**.** $\{X_t\}$ is a moving average process of order q (MA(q) process) if

$$X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}, \tag{13.1}$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and $\theta_1, \ldots, \theta_q$ are constants.

Remark. If $\{Z_t\}$ is iid noise, then (13.1) defines a stationary time series that is strictly stationary. It follows also that $\{X_t\}$ is q-dependent, i.e., that X_s and X_t are independent whenever $|t-s| > q$.

Remark. We say that a stationary time series is q-correlated if $\gamma(h) = 0$ whenever $|h| > q$. A white noise sequence is then 0-correlated, while the MA(1) process is 1-correlated.

The importance of MA(q) processes derives from the fact that every q-correlated process is an MA(q) process, i.e., if $\{X_t\}$ is a stationary q-correlated time series with mean 0, then it can be represented as the MA(q) process in (13.1).

Definition 13.1.7 (**AR(p)** process)**.** $\{X_t\}$ is an autoregressive process of order p if

$$X_t = \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + Z_t \tag{13.2}$$

where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and $\phi_1, \ldots, \phi_p$ are constants.

Example. Figure 13.1 shows different MA(q) and AR(p) processes.

13.2 Linear Processes

The class of linear time series models, which includes the class of autoregressive moving average (ARMA) models (see Chapter 14), provides a general framework for studying stationary processes. In fact, every weakly stationary process is either a linear process or can be transformed to a linear process by subtracting a deterministic component. This result is known as Wold's decomposition (see Brockwell and Davis (1991), pp. 187-191). Therefore we cite some results of the theory of linear processes.

Definition 13.2.1. The time series $\{X_t\}$ is a linear process if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}, \tag{13.3}$$

for all t, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$ and $\{\psi_j\}$ is a sequence of constants with $\sum_{-\infty}^{\infty} |\psi_j| < \infty$.

Remark. In terms of the backward shift operator B, the linear process (13.3) can be written more compactly as

$$X_t = \psi(B) Z_t,$$

where $\psi(B) = \sum_{j=-\infty}^{\infty} \psi_j B^j$.

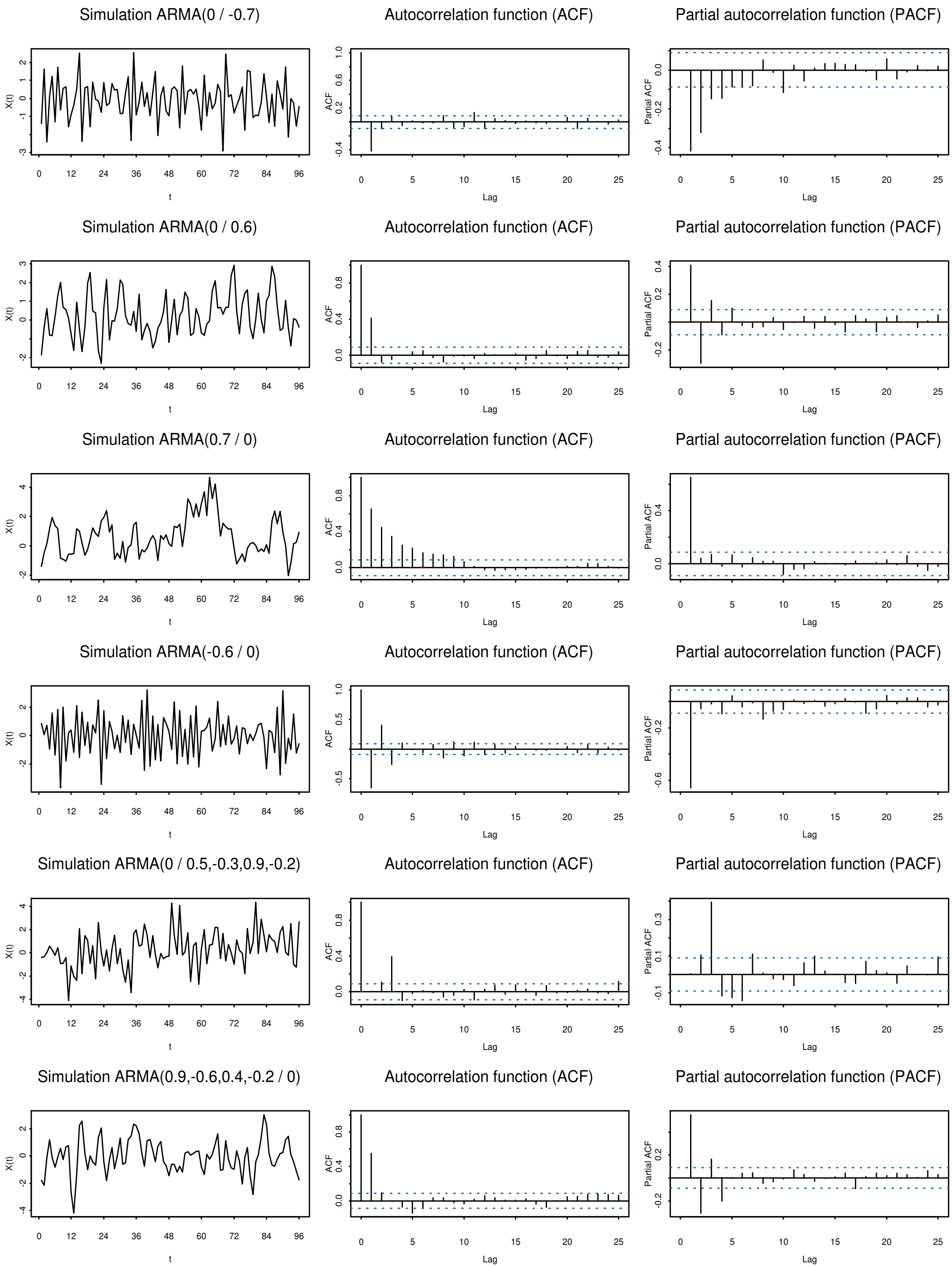

Figure 13.1: Simulations of different MA(q) and AR(p) processes. The left column shows an excerpt ($m = 96$) of the whole time series ($n = 480$).

Remark. A linear process is called a moving average or MA(∞) if $\psi_j = 0$ for all $j < 0$, i.e., if

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}.$$

Proposition 13.2.2. *Let $\{Y_t\}$ be a stationary time series with mean 0 and covariance function γ_Y. If $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, then the time series*

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Y_{t-j} = \psi(B) Y_t$$

is stationary with mean 0 and autocovariance function

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h + k - j). \tag{13.4}$$

In the special case where $\{X_t\}$ is a linear process,

$$\gamma_X(h) = \sigma^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h}. \tag{13.5}$$

Proof. Since $\mathrm{E}Y_t = 0$, we have

$$\begin{aligned}
\mathrm{E}X_t &= 0, \\
\mathrm{E}(X_{t+h}X_t) &= \mathrm{E}\left[\left(\sum_{j=-\infty}^{\infty} \psi_j Y_{t+h-j}\right)\left(\sum_{k=-\infty}^{\infty} \psi_k Y_{t-k}\right)\right] \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \mathrm{E}(Y_{t+h-j} Y_{t-k}) \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Y(h - j + k),
\end{aligned}$$

which shows that $\{X_t\}$ is stationary with covariance function (13.4). Finally, if $\{Y_t\}$ is the white noise sequence $\{Z_t\}$ in (13.3), then $\gamma_Y(h - j + k) = \sigma^2$ if $k = j - h$ and 0 otherwise, from which (13.5) follows. □

Example. Consider the MA(q) process in (13.1). We find $\mathrm{E}X_t = 0$ and $\mathrm{E}X_t^2 = \sigma^2 \sum_{j=0}^{q} \theta_j^2$ with $\theta_0 = 1$ and with Proposition 13.2.2 we get

$$\gamma(h) = \begin{cases} \sigma^2 \sum_{j=0}^{q-|h|} \theta_j \theta_{j+|h|}, & \text{if } |h| \le q, \\ 0, & \text{if } |h| > q. \end{cases}$$

Example. Consider the AR(1) equation

$$X_t = \phi X_{t-1} + Z_t, \quad t = 0, \pm 1, \pm 2, \ldots$$

(see page 12-12). Although the series is first observed at time $t = 0$, the process is regarded as having started at some time in the remote past. Substituting for lagged values of X_t gives

$$X_t = \sum_{j=0}^{J-1} \phi^j Z_{t-j} + \phi^J X_{t-J}. \tag{13.6}$$

The right hand side consists of two parts, the first of which is a moving average of lagged values of the white noise variable driving the process. The second part depends on the value of X_t at time $t - J$. Taking expectations and treating X_{t-J} as a fixed number yields

$$\mathrm{E}(X_t) = \mathrm{E}\left(\sum_{j=0}^{J-1} \phi^j Z_{t-j}\right) + \mathrm{E}(\phi^J X_{t-J}) = \phi^J X_{t-J}.$$

If $|\phi| \geq 1$, the mean value of the process depends on the starting value, X_{t-J}. Expression (13.6) therefore contains a deterministic component and a knowledge of X_{t-J} enables non-trivial prediction to be made for future values of the series. If, on the other hand, $|\phi| < 1$, this deterministic component is negligible if J is large. As $J \to \infty$, it effectively disappears and so if the process is regarded as having started at some point in the remote past, it is quite legitimate to write (13.6) in the form

$$X_t = \sum_{j=0}^{\infty} \phi^j Z_{t-j}, \quad t = 0, \ldots, T.$$

Since $\sum_{j=0}^{\infty} |\phi|^j < \infty$ it follows from Proposition 13.2.2 that the AR(1) process is stationary with mean 0 if $|\phi| < 1$ and the autocovariance function is given by

$$\gamma_X(h) = \sigma^2 \sum_{j=0}^{\infty} \phi^j \phi^{j+h} = \sigma^2 \frac{\phi^h}{1 - \phi^2}$$

for $h > 0$.