Table A.15 Test for Independence of p Variables

Upper percentage points for

$$u' = -\left(\nu - \frac{2p+5}{6}\right)\ln\left(\frac{|\mathbf{S}|}{s_{11}\cdots s_{pp}}\right) = -\left(\nu - \frac{2p+5}{6}\right)\ln|\mathbf{R}|,$$

where ν is the degrees of freedom of **S** or **R**. Reject if u' is greater than table value. The χ^2_α values are shown for comparison, since u' is approximately χ^2 distributed with $f = \frac{1}{2}p(p-1)$ degrees of freedom.

n	$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 7$	$p = 8$	$p = 9$	$p = 10$
				$\alpha = 0.05$				
4	8.020							
5	7.834	15.22						
6	7.814	13.47	24.01					
7	7.811	13.03	20.44	34.30				
8	7.811	12.85	19.45	28.75	46.05			
9	7.811	12.76	19.02	27.11	38.41	59.25		
10	7.812	12.71	18.80	26.37	36.03	49.42	73.79	
11	7.812	12.68	18.67	25.96	34.91	46.22	61.76	89.92
12	7.813	12.66	18.58	25.71	34.28	44.67	57.68	75.45
13	7.813	12.65	18.52	25.55	33.89	43.78	55.65	70.43
14	7.813	12.64	18.48	25.44	33.63	43.21	54.46	67.87
15	7.813	12.63	18.45	25.36	33.44	42.82	53.69	66.34
16	7.814	12.62	18.43	25.30	33.31	42.55	53.15	65.33
17	7.814	12.62	18.41	25.25	33.20	42.34	52.77	64.63
18	7.814	12.62	18.40	25.21	33.12	42.19	52.48	64.12
19	7.814	12.61	18.38	25.19	33.06	42.06	52.26	63.73
20	7.814	12.61	18.37	25.16	33.01	41.97	52.08	63.43
$\chi^2_{0.05}$	7.815	12.59	18.31	25.00	32.67	41.34	51.00	61.66
				$\alpha = 0.01$				
4	11.79							
5	11.41	21.18						
6	11.36	18.27	32.16					
7	11.34	17.54	26.50	44.65				
8	11.34	17.24	24.95	36.09	58.61			
9	11.34	17.10	24.29	33.63	47.05	74.01		
10	11.34	17.01	23.95	32.54	43.59	59.36	90.87	
11	11.34	16.96	23.75	31.95	42.00	54.83	73.03	109.53
12	11.34	16.93	23.62	31.60	41.13	52.70	67.37	88.05
13	11.34	16.90	23.53	31.36	40.59	51.49	64.64	81.20
14	11.34	16.89	23.47	31.20	40.23	50.73	63.06	77.83
15	11.34	16.87	23.42	31.09	39.97	50.22	62.05	75.84
16	11.34	16.86	23.39	31.00	39.79	49.85	61.36	74.56
17	11.34	16.86	23.36	30.94	39.65	49.59	60.86	73.66
18	11.34	16.85	23.34	30.88	39.54	49.38	60.49	73.01
19	11.34	16.85	23.32	30.84	39.46	49.22	60.21	72.52
20	11.34	16.84	23.31	30.81	39.39	49.09	59.99	72.15
$\chi^2_{0.01}$	11.34	16.81	23.21	30.58	38.93	48.28	58.57	69.92

APPENDIX B

Answers and Hints to Problems

CHAPTER 2

2.1 **(a)** $\mathbf{A}+\mathbf{B} = \begin{pmatrix} 7 & 0 & 7 \\ 13 & 14 & 3 \end{pmatrix}, \quad \mathbf{A}-\mathbf{B} = \begin{pmatrix} 1 & 4 & -1 \\ 1 & -4 & 13 \end{pmatrix}$

(b) $\mathbf{A}'\mathbf{A} = \begin{pmatrix} 65 & 43 & 68 \\ 43 & 29 & 46 \\ 68 & 46 & 73 \end{pmatrix}, \quad \mathbf{A}\mathbf{A}' = \begin{pmatrix} 29 & 62 \\ 62 & 138 \end{pmatrix}$

2.2 **(a)** $(\mathbf{A}+\mathbf{B})' = \begin{pmatrix} 7 & 13 \\ 0 & 14 \\ 7 & 3 \end{pmatrix}, \quad \mathbf{A}'+\mathbf{B}' = \begin{pmatrix} 7 & 13 \\ 0 & 14 \\ 7 & 3 \end{pmatrix}$

(b) $\mathbf{A}' = \begin{pmatrix} 4 & 7 \\ 2 & 5 \\ 3 & 8 \end{pmatrix}, \quad (\mathbf{A}')' = \begin{pmatrix} 4 & 2 & 3 \\ 7 & 5 & 8 \end{pmatrix} = \mathbf{A}$

2.3 **(a)** $\mathbf{AB} = \begin{pmatrix} 5 & 15 \\ 3 & -5 \end{pmatrix}, \quad \mathbf{BA} = \begin{pmatrix} 2 & 6 \\ 11 & -2 \end{pmatrix}$

(b) $|\mathbf{AB}| = -70, \quad |\mathbf{A}| = -7, \quad |\mathbf{B}| = 10$

2.4 **(a)** $\mathbf{A}+\mathbf{B} = \begin{pmatrix} 3 & 3 \\ 3 & 4 \end{pmatrix}, \quad \text{tr}(\mathbf{A}+\mathbf{B}) = 7$

(b) $\text{tr}(\mathbf{A}) = 0, \quad \text{tr}(\mathbf{B}) = 7$

2.5 **(a)** $\mathbf{AB} = \begin{pmatrix} 4 & 1 \\ 3 & -3 \end{pmatrix}, \quad \mathbf{BA} = \begin{pmatrix} -1 & 8 & 7 \\ 2 & 4 & 6 \\ 1 & -3 & -2 \end{pmatrix}$

(b) $\text{tr}(\mathbf{AB}) = 1, \quad \text{tr}(\mathbf{BA}) = 1$

2.6 **(b)** $\mathbf{x} = (1 \quad 1 \quad -1)'$

2.7 (a) $\mathbf{Bx} = (13, 6, 9)'$ (b) $\mathbf{y'B} = (25, -1, 17)$ (c) $\mathbf{x'Ax} = 16$
(d) $\mathbf{x'Ay} = 43$ (e) $\mathbf{x'x} = 6$ (f) $\mathbf{x'y} = 3$

(g) $\mathbf{xx'} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix}$ (h) $\mathbf{xy'} = \begin{pmatrix} 3 & 2 & 1 \\ -3 & -2 & -1 \\ 6 & 4 & 2 \end{pmatrix}$

(i) $\mathbf{B'B} = \begin{pmatrix} 62 & 7 & 22 \\ 7 & 14 & 7 \\ 22 & 7 & 41 \end{pmatrix}$

2.8 (a) $\mathbf{x} + \mathbf{y} = (4, 1, 3)'$, $\mathbf{x} - \mathbf{y} = (-2, -3, 1)'$, (b) $(\mathbf{x} - \mathbf{y})'\mathbf{A}(\mathbf{x} - \mathbf{y}) = -31$

2.9 $$\mathbf{Bx} = \mathbf{b}_1 x_1 + \mathbf{b}_2 x_2 + \mathbf{b}_3 x_3 = (1)\begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix} + (-1)\begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + (2)\begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 13 \\ 6 \\ 9 \end{pmatrix}$$

2.10 (a) $(\mathbf{AB})' = \begin{pmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{pmatrix}$, $\mathbf{B'A'} = \begin{pmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{pmatrix}$ (c) $|\mathbf{A}| = 5$

2.11 (a) $\mathbf{a'b} = 5$, $(\mathbf{a'b})^2 = 25$ (b) $\mathbf{bb'} = \begin{pmatrix} 4 & 2 & 6 \\ 2 & 1 & 3 \\ 6 & 3 & 9 \end{pmatrix}$, $\mathbf{a'(bb')a} = 25$

2.12 $\mathbf{DA} = \begin{pmatrix} a & 2a & 3a \\ 4b & 5b & 6b \\ 7c & 8c & 9c \end{pmatrix}$, $\mathbf{AD} = \begin{pmatrix} a & 2b & 3c \\ 4a & 5b & 6c \\ 7a & 8b & 9c \end{pmatrix}$,

$\mathbf{DAD} = \begin{pmatrix} a^2 & 2ab & 3ac \\ 4ab & 5b^2 & 6bc \\ 7ac & 8bc & 9c^2 \end{pmatrix}$

2.13 $$\mathbf{AB} = \left[\begin{array}{ccc|c} 8 & 9 & 5 & 6 \\ 7 & 5 & 5 & 4 \\ \hline 3 & 4 & 2 & 2 \end{array}\right]$$

2.14 $\mathbf{AB} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$, $\mathbf{CB} = \begin{pmatrix} 3 & 5 \\ 1 & 4 \end{pmatrix}$

2.15 (a) $\text{tr}(\mathbf{A}) = 5$, $\text{tr}(\mathbf{B}) = 5$

(b) $\mathbf{A} + \mathbf{B} = \begin{pmatrix} 6 & 4 & 5 \\ 2 & -2 & 1 \\ 4 & 9 & 6 \end{pmatrix}$, $\text{tr}(\mathbf{A} + \mathbf{B}) = 10$

(c) $|\mathbf{A}| = 0$, $|\mathbf{B}| = 2$

(d) $\mathbf{AB} = \begin{pmatrix} 9 & 12 & 17 \\ 3 & -1 & 5 \\ 6 & 13 & 12 \end{pmatrix}$, $\det(\mathbf{AB}) = 0$

2.16 (a) $|\mathbf{A}| = 36$ (b) $\mathbf{T} = \begin{pmatrix} 1.7321 & 2.3094 & 1.7321 \\ 0 & 1.6330 & 1.2247 \\ 0 & 0 & 2.1213 \end{pmatrix}$

2.17 (a) $\det(\mathbf{A}) = 1$ (b) $\mathbf{T} = \begin{pmatrix} 1.7321 & -2.8868 & -.5774 \\ 0 & 2.1602 & -.7715 \\ 0 & 0 & .2673 \end{pmatrix}$

2.18 (a) $\mathbf{C} = \begin{pmatrix} .4082 & -.5774 & .7071 \\ .8165 & .5774 & .0000 \\ .4082 & -.5774 & -.7071 \end{pmatrix}$

2.19 (a) Eigenvalues: $2, 1, -1$

Eigenvectors: $\begin{pmatrix} .3015 \\ .9045 \\ .3015 \end{pmatrix}, \begin{pmatrix} .7999 \\ .5368 \\ .2684 \end{pmatrix}, \begin{pmatrix} .7071 \\ 0 \\ .7071 \end{pmatrix}$

(b) $\text{tr}(\mathbf{A}) = 2$, $|\mathbf{A}| = -2$

2.20 (a) $\mathbf{C} = \begin{pmatrix} .0000 & .5774 & -.8165 \\ -.7071 & -.5774 & -.4082 \\ .7071 & -.5774 & -.4082 \end{pmatrix}$,

(b) $\mathbf{C'AC} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(c) $\mathbf{CDC'} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} = \mathbf{A}$

2.21 Eigenvalues: $1, 3$, $\mathbf{C} = \begin{pmatrix} -.7071 & -.7071 \\ -.7071 & .7071 \end{pmatrix}$,

$\mathbf{A}^{1/2} = \mathbf{CD}^{1/2}\mathbf{C}' = \begin{pmatrix} 1.3660 & -.3660 \\ -.3660 & 1.3660 \end{pmatrix}$

2.22 (a) $\mathbf{j'a} = (1)a_1 + (1)a_2 + \cdots + (1)a_n = \sum_i a_i = \mathbf{a'j}$

$$\text{(b)}\quad \mathbf{Aj} = \begin{bmatrix} (1)a_{11} + (1)a_{12} + \cdots + (1)a_{1p} \\ (1)a_{21} + (1)a_{22} + \cdots + (1)a_{2p} \\ \vdots \\ (1)a_{n1} + (1)a_{n2} + \cdots + (1)a_{np} \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j} \\ \sum_j a_{2j} \\ \vdots \\ \sum_j a_{nj} \end{bmatrix}$$

$$\begin{aligned} \text{(c)}\quad \mathbf{j}'\mathbf{A} &= [(1)a_{11} + (1)a_{21} + \cdots + (1)a_{n1}, \ldots, (1)a_{1p} \\ &\quad + (1)a_{2p} + \cdots + (1)a_{np}] \\ &= (\textstyle\sum_i a_{i1}, \sum_i a_{i2}, \ldots, \sum_i a_{ip}) \end{aligned}$$

2.23
$$\begin{aligned} (\mathbf{x} - \mathbf{y})'(\mathbf{x} - \mathbf{y}) &= (\mathbf{x}' - \mathbf{y}')(\mathbf{x} - \mathbf{y}) \\ &= \mathbf{x}'\mathbf{x} - \mathbf{x}'\mathbf{y} - \mathbf{y}'\mathbf{x} + \mathbf{y}'\mathbf{y} \\ &= \mathbf{x}'\mathbf{x} - 2\mathbf{x}'\mathbf{y} + \mathbf{y}'\mathbf{y} \end{aligned}$$

2.24 By (2.27), $(\mathbf{A}'\mathbf{A})' = \mathbf{A}'(\mathbf{A}')'$. By (2.6), $(\mathbf{A}')' = \mathbf{A}$. Thus, $(\mathbf{A}'\mathbf{A})' = \mathbf{A}'\mathbf{A}$.

2.25
$$\begin{aligned} \text{(a)}\quad \textstyle\sum_i \mathbf{a}'\mathbf{x}_i &= \mathbf{a}'\mathbf{x}_1 + \mathbf{a}'\mathbf{x}_2 + \cdots + \mathbf{a}'\mathbf{x}_n \\ &= \mathbf{a}'(\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n) \quad [\text{by } (2.21)] \\ &= \mathbf{a}'\textstyle\sum_i \mathbf{x}_i \end{aligned}$$

$$\begin{aligned} \text{(b)}\quad \textstyle\sum_i \mathbf{A}\mathbf{x}_i &= \mathbf{A}\mathbf{x}_1 + \mathbf{A}\mathbf{x}_2 + \cdots + \mathbf{A}\mathbf{x}_n \\ &= \mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n) \quad [\text{by } (2.21)] \\ &= \mathbf{A}\textstyle\sum_i \mathbf{x}_i \end{aligned}$$

$$\begin{aligned} \text{(c)}\quad \textstyle\sum_i (\mathbf{a}'\mathbf{x}_i)^2 &= \textstyle\sum_i \mathbf{a}'(\mathbf{x}_i\mathbf{x}_i')\mathbf{a} \quad [\text{by } (2.40)] \\ &= \mathbf{a}'(\textstyle\sum_i \mathbf{x}_i\mathbf{x}_i')\mathbf{a} \quad [\text{by } (2.29)] \end{aligned}$$

$$\text{(d)}\quad \textstyle\sum_i \mathbf{A}\mathbf{x}_i(\mathbf{A}\mathbf{x}_i)' = \sum_i \mathbf{A}\mathbf{x}_i\mathbf{x}_i'\mathbf{A}' = \mathbf{A}(\sum_i \mathbf{x}_i\mathbf{x}_i')\mathbf{A}'$$

2.26
$$\text{(a)}\quad \mathbf{A}\mathbf{x} = \begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{pmatrix}\mathbf{x} = \begin{pmatrix} \mathbf{a}_1'\mathbf{x} \\ \mathbf{a}_2'\mathbf{x} \end{pmatrix}$$

$$\begin{aligned} \text{(b)}\quad \mathbf{A}'\mathbf{S}\mathbf{A} &= \begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{pmatrix}\mathbf{S}(\mathbf{a}_1, \mathbf{a}_2) = \begin{pmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \end{pmatrix}(\mathbf{S}\mathbf{a}_1, \mathbf{S}\mathbf{a}_2) \quad [\text{by } (2.47)] \\ &= \begin{pmatrix} \mathbf{a}_1'\mathbf{S}\mathbf{a}_1 & \mathbf{a}_1'\mathbf{S}\mathbf{a}_2 \\ \mathbf{a}_2'\mathbf{S}\mathbf{a}_1 & \mathbf{a}_2'\mathbf{S}\mathbf{a}_2 \end{pmatrix} \end{aligned}$$

2.27 If $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \\ \vdots \\ \mathbf{a}_n' \end{bmatrix}$, then by (2.63), $\mathbf{A}' = (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)$ and

$$\begin{aligned} \mathbf{A}'\mathbf{A} &= (\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n)\begin{bmatrix} \mathbf{a}_1' \\ \mathbf{a}_2' \\ \vdots \\ \mathbf{a}_n' \end{bmatrix} \\ &= \mathbf{a}_1\mathbf{a}_1' + \mathbf{a}_2\mathbf{a}_2' + \cdots + \mathbf{a}_n\mathbf{a}_n' \quad [\text{by } (2.60)] \end{aligned}$$

2.28
$$\begin{aligned} &\frac{1}{b}\begin{pmatrix} b\mathbf{A}_{11}^{-1} + \mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} & -\mathbf{A}_{11}^{-1}\mathbf{a}_{12} \\ -\mathbf{a}_{12}'\mathbf{A}_{11}^{-1} & 1 \end{pmatrix}\begin{pmatrix} \mathbf{A}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{12}' & a_{22} \end{pmatrix} \\ &= \frac{1}{b}\begin{pmatrix} b\mathbf{I} + \mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}_{12}' - \mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}_{12}' & b\mathbf{A}_{11}^{-1}\mathbf{a}_{12} + \mathbf{A}_{11}^{-1}\mathbf{a}_{12}\mathbf{a}_{12}'\mathbf{A}_{11}^{-1}\mathbf{a}_{12} - \mathbf{A}_{11}^{-1}\mathbf{a}_{12}a_{22} \\ -\mathbf{a}_{12}' + \mathbf{a}_{12}' & -\mathbf{a}_{12}'\mathbf{A}_{11}^{-1}\mathbf{a}_{12} + a_{22} \end{pmatrix} \\ &= \frac{1}{b}\begin{pmatrix} b\mathbf{I} & \mathbf{0} \\ \mathbf{0}' & b \end{pmatrix} \quad \text{where } b = a_{22} - \mathbf{a}_{12}'\mathbf{A}_{11}^{-1}\mathbf{a}_{12} \\ &= \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & 1 \end{pmatrix} \end{aligned}$$

2.29
$$\begin{aligned} &(\mathbf{B} + \mathbf{c}\mathbf{c}')\left(\mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}}{1 + \mathbf{c}'\mathbf{B}^{-1}\mathbf{c}}\right) \\ &= \mathbf{I} - \frac{\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}}{1 + \mathbf{c}'\mathbf{B}^{-1}\mathbf{c}} + \mathbf{c}\mathbf{c}'\mathbf{B}^{-1} - \frac{\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}\mathbf{c}\mathbf{c}'\mathbf{B}^{-1}}{1 + \mathbf{c}'\mathbf{B}^{-1}\mathbf{c}} \quad [\text{by } (2.26)] \\ &= \mathbf{I} - \mathbf{c}\mathbf{c}'\mathbf{B}^{-1}\left(\frac{1 + \mathbf{c}'\mathbf{B}^{-1}\mathbf{c}}{1 + \mathbf{c}'\mathbf{B}^{-1}\mathbf{c}}\right) + \mathbf{c}\mathbf{c}'\mathbf{B}^{-1} = \mathbf{I} \end{aligned}$$

2.30
$$\begin{aligned} &\mathbf{A}\mathbf{A}^{-1} = \mathbf{I} \\ &|\mathbf{A}\mathbf{A}^{-1}| = |\mathbf{I}| \\ &|\mathbf{A}||\mathbf{A}^{-1}| = 1 \quad [\text{by } (2.83)] \\ &|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} \end{aligned}$$

2.31 In (2.86) and (2.87), let $\mathbf{A}_{11} = \mathbf{B}$, $\mathbf{A}_{12} = \mathbf{c}$, $\mathbf{A}_{21} = -\mathbf{c}'$, and $\mathbf{A}_{22} = 1$. Then equate the right-hand sides of (2.86) and (2.87) to obtain (2.109).

2.32 Show that $|\mathbf{C}| \neq 0$ by taking the determinant of both sides of $\mathbf{C}'\mathbf{C} = \mathbf{I}$. Thus $\mathbf{C}$ is nonsingular and $\mathbf{C}^{-1}$ exists. Multiply $\mathbf{C}'\mathbf{C} = \mathbf{I}$ on the right by $\mathbf{C}^{-1}$ and on the left by $\mathbf{C}$.

2.33 Multiply $\mathbf{AB}\mathbf{x} = \lambda\mathbf{x}$ on the left by $\mathbf{B}$. Then λ is an eigenvalue of $\mathbf{BA}$ and $\mathbf{B}\mathbf{x}$ is an eigenvector.

2.34 (a) $(\mathbf{A}^{1/2})^2 = (\mathbf{C}\mathbf{D}^{1/2}\mathbf{C}')^2 = \mathbf{C}\mathbf{D}^{1/2}\mathbf{C}'\mathbf{C}\mathbf{D}^{1/2}\mathbf{C}'$

$$= \mathbf{CDC}' \quad [\text{by } (2.92)]$$
$$= \mathbf{A} \quad [\text{by } (2.100)]$$

(b) By (2.105), $\mathbf{A}^{1/2}\mathbf{A}^{1/2} = \mathbf{A}$. By (2.83),

$$\begin{aligned} |\mathbf{A}^{1/2}\mathbf{A}^{1/2}| &= |\mathbf{A}| \\ |\mathbf{A}^{1/2}||\mathbf{A}^{1/2}| &= |\mathbf{A}| \\ |\mathbf{A}^{1/2}|^2 &= |\mathbf{A}| \end{aligned}$$

(c) Since **A** is positive definite, we have, from part (b), $|\mathbf{A}^{1/2}| = |\mathbf{A}|^{1/2}$.

2.35 $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
$(\mathbf{A}^{-1}\mathbf{A})' = \mathbf{I}' = \mathbf{I}$
$\mathbf{A}'(\mathbf{A}^{-1})' = \mathbf{I}$
$(\mathbf{A}')^{-1}\mathbf{A}'(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}\mathbf{I} = (\mathbf{A}')^{-1}$
$(\mathbf{A}^{-1})' = (\mathbf{A}')^{-1}$

CHAPTER 3

3.1 $\bar{z} = \sum_{i=1}^n z_i/n = \sum_i ay_i/n = (ay_1 + \cdots + ay_n)/n$. Now factor a out of the sum.

3.2 The numerator of s_z^2 is $\sum_{i=1}^n (z_i - \bar{z})^2 = \sum_i (ay_i - a\bar{y})^2 = \sum_i [a(y_i - \bar{y})]^2$.

3.3 $\bar{x} = 4, \bar{y} = 4$:

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})(y-\bar{y})$
2	2	−2	−2	4
2	4	−2	0	0
2	6	−2	2	−4
4	2	0	−2	0
4	4	0	0	0
4	6	0	2	0
6	2	2	−2	−4
6	4	2	0	0
6	6	2	2	4
				Sum = 0

3.4 $$\mathbf{x} - \bar{x}\mathbf{j} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \bar{x}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} \bar{x} \\ \bar{x} \\ \vdots \\ \bar{x} \end{bmatrix} = \begin{bmatrix} x_1 - \bar{x} \\ x_2 - \bar{x} \\ \vdots \\ x_n - \bar{x} \end{bmatrix}$$

3.5 $$\mathbf{y}_i - \bar{\mathbf{y}} = \begin{pmatrix} y_{i1} \\ y_{i2} \\ y_{i3} \end{pmatrix} - \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \\ \bar{y}_3 \end{pmatrix} = \begin{pmatrix} y_{i1} - \bar{y}_1 \\ y_{i2} - \bar{y}_2 \\ y_{i3} - \bar{y}_3 \end{pmatrix}$$

$$\sum_{i=1}^n (\mathbf{y}_i - \bar{\mathbf{y}})(\mathbf{y}_i - \bar{\mathbf{y}})' = \sum_{i=1}^n \begin{pmatrix} y_{i1} - \bar{y}_1 \\ y_{i2} - \bar{y}_2 \\ y_{i3} - \bar{y}_3 \end{pmatrix} (y_{i1} - \bar{y}_1, y_{i2} - \bar{y}_2, y_{i3} - \bar{y}_3)$$

$$= \sum_{i=1}^n \begin{pmatrix} (y_{i1} - \bar{y}_1)^2 & (y_{i1} - \bar{y}_1)(y_{i2} - \bar{y}_2) & (y_{i1} - \bar{y}_1)(y_{i3} - \bar{y}_3) \\ (y_{i2} - \bar{y}_2)(y_{i1} - \bar{y}_1) & (y_{i2} - \bar{y}_2)^2 & (y_{i2} - \bar{y}_2)(y_{i3} - \bar{y}_3) \\ (y_{i3} - \bar{y}_3)(y_{i1} - \bar{y}_1) & (y_{i3} - \bar{y}_3)(y_{i2} - \bar{y}_2) & (y_{i3} - \bar{y}_3)^2 \end{pmatrix}$$

3.6 $\bar{z} = \sum_{i=1}^n z_i/n = \sum_i \mathbf{a}'\mathbf{y}_i/n = (\mathbf{a}'\mathbf{y}_1 + \cdots + \mathbf{a}'\mathbf{y}_n)/n$. Now factor out $\mathbf{a}'$ on the left. See also (2.42).

3.7 The numerator of s_z^2 is $\sum_{i=1}^n (z_i - \bar{z})^2 = \sum_i (\mathbf{a}'\mathbf{y}_i - \mathbf{a}'\bar{\mathbf{y}})^2 = \sum_i (\mathbf{a}'\mathbf{y}_i - \mathbf{a}'\bar{\mathbf{y}})(\mathbf{a}'\mathbf{y}_i - \mathbf{a}'\bar{\mathbf{y}})$. The scalar $\mathbf{a}'\mathbf{y}_i$ is equal to its transpose, as in (2.39). Thus $\mathbf{a}'\mathbf{y}_i = (\mathbf{a}'\mathbf{y}_i)' = \mathbf{y}_i'\mathbf{a}$, and $\sum_i (\mathbf{a}'\mathbf{y}_i - \mathbf{a}'\bar{\mathbf{y}})(\mathbf{a}'\mathbf{y}_i - \mathbf{a}'\bar{\mathbf{y}}) = \sum_i (\mathbf{a}'\mathbf{y}_i - \mathbf{a}'\bar{\mathbf{y}})(\mathbf{y}_i'\mathbf{a} - \bar{\mathbf{y}}'\mathbf{a})$. By (2.22) and (2.24), this becomes $\sum_i \mathbf{a}'(\mathbf{y}_i - \bar{\mathbf{y}})(\mathbf{y}_i - \bar{\mathbf{y}})'\mathbf{a}$. Now factor out $\mathbf{a}'$ on the left and $\mathbf{a}$ on the right. See also (2.44).

3.8 See problem 2.26(b).

3.9 In the discussion preceding (3.60), we have $\mathbf{ASA}'$ in the form

$$\mathbf{ASA}' = \begin{bmatrix} \mathbf{a}_1'\mathbf{Sa}_1 & \mathbf{a}_1'\mathbf{Sa}_2 & \cdots & \mathbf{a}_1'\mathbf{Sa}_k \\ \mathbf{a}_2'\mathbf{Sa}_1 & \mathbf{a}_2'\mathbf{Sa}_2 & \cdots & \mathbf{a}_2'\mathbf{Sa}_k \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_k'\mathbf{Sa}_1 & \mathbf{a}_k'\mathbf{Sa}_2 & \cdots & \mathbf{a}_k'\mathbf{Sa}_k \end{bmatrix},$$

from which the result follows immediately.

3.10 $$\begin{aligned} \text{cov}(\mathbf{z}) &= \text{cov}[(\boldsymbol{\Sigma}^{1/2})^{-1}\bar{\mathbf{y}} - (\boldsymbol{\Sigma}^{1/2})^{-1}\boldsymbol{\mu}] \\ &= (\boldsymbol{\Sigma}^{1/2})^{-1}\,\text{cov}(\bar{\mathbf{y}})[(\boldsymbol{\Sigma}^{1/2})^{-1}]' \quad [\text{by } (3.71)] \\ &= (\boldsymbol{\Sigma}^{1/2})^{-1}\left(\frac{\boldsymbol{\Sigma}}{n}\right)(\boldsymbol{\Sigma}^{1/2})^{-1} \end{aligned}$$